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EXISTENCE OF ASYMPTOTIC SPEED FOR LEVEL SET EQUATIONS WITH SOURCE TERM

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1. Introduction

The main purpose of this proceeding is to briefly and simply describe some results in [4], which have recently been obtained jointly with Y. Giga, T. Ohtsuka, H. V. Tran, on asymptotic speed of solutions to level set equations with source term.

We are concerned with a quasi-nonlinear, possibly degenerate parabolic partial differential equation (PDE) of the form:

$$\text{(C)} \qquad \begin{cases} u_t - \left(\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + 1\right)|Du| = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^n, \end{cases}$$

where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a unknown function, and u_t, Du and div denote the time derivative, the spatial gradient and divergence, respectively. Here, the function $f : \mathbb{R}^n \to [0,\infty)$, which is called a *source term* in the paper, is a given function. We *always* assume that

$$f \in C_c^1(\mathbb{R}^n) \text{ and there exists } R_0 > 0 \text{ such that } \operatorname{supp}(f) \subset B(0,R_0), \tag{1.1}$$

and $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a bounded uniformly continuous function on $\mathbb{R}^n$.

We study the large time average of u as $t \to \infty$, that is,

$$\lim_{t\to\infty} \frac{u(x,t)}{t} \quad \text{for each given } x \in \mathbb{R}^n. \tag{1.2}$$

We call the limit in (1.2) the *asymptotic speed* for (C) if it exists. This is a quite standard question in nonlinear PDEs and it has been studying a lot recently in the context of periodic homogenization theory, large time behavior, crystal growth, etc.

In this proceeding, we present a simple way to prove the existence of the asymptotic speed, which is quite robust. We can apply it to more general fully nonlinear PDEs. See [4] for details.

This note is organized as following: in Section 2, we derive (C) by using a kind of birth and spread model. Section 3 is devoted to give Lipschitz estimates on solutions to (C),

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and we give a main result on the asymptotic speed in Section 4. Finally, in Section 5, we discuss some of partial results of qualitative properties of the asymptotic speed.

2. Birth and Spread Model

In this section, we derive (C) by using the double-step method, which is considered as a kind of birth and spread model in the theory of crystal growth (see [6, Section 2.6] for details).

Consider two initial-value problems:

$$\text{(N)} \quad \begin{cases} v_t = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(\cdot,0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

and

$$\text{(P)} \quad \begin{cases} w_t = \left(\operatorname{div}\left(\dfrac{Dw}{|Dw|}\right) + 1\right)|Dw| & \text{in } \mathbb{R}^n \times (0,\infty), \\ w(\cdot,0) = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

We call (N) and (P) the *nucleation problem* and the *propagation problem*, respectively. Define the operators $S_1(t), S_2(t) : \mathrm{Lip}\,(\mathbb{R}^n) \to \mathrm{Lip}\,(\mathbb{R}^n)$, repectively, by

$$S_1(t)[u_0] := u_0 + f(\cdot)t, \quad \text{and} \quad S_2(t)[u_0] := w(\cdot,t), \tag{2.1}$$

where w is the unique viscosity solution of (P). For $x \in \mathbb{R}^n, \tau > 0, i \in \mathbb{N}$, set

$$U^\tau(x, i\tau) := S_1(\tau)\big(S_2(\tau)S_1(\tau)\big)^i[u_0]. \tag{2.2}$$

We call $U^\tau(x,i\tau)$ the *Trotter-Kato product formula.*

There is a nice general framework for this formula in the theory of viscosity solutions established by Barles and Souganidis in [2]. If we apply the framework in [2], then we can prove that

$$\lim_{i\to\infty} U^\tau(x,i\tau) = u(x,t) \quad \text{locally uniformly for } x \in \mathbb{R}^n, \tag{2.3}$$

where u is the viscosity solution of (C).

In light of this, the behavior of $u(x,t)/t$ as $t \to \infty$ can be consider as the behavior of

$$\lim_{t\to\infty}\left(\lim_{\substack{\tau\to 0\\ i\tau = t}} \frac{U^\tau(x,i\tau)}{i\tau}\right). \tag{2.4}$$

The advantage of considering $U^\tau(x,i\tau)$ lies in the fact that its graph is a pyramid of finite number of steps. The double-step method can then be described in a geometrical way as follows:

- **(N)** At each nucleation step, we drop from above an amount of $\tau f(x)$ crystal down to the pyramid with the assumption that the crystals are not sticky;
- **(P)** At each propagation step, each layer of the pyramid evolves under a forced mean curvature flow ($V = \kappa + 1$), where V is the outward normal velocity and κ is its mean curvature in the direction of the outer normal vector.

Let us emphasize that, in general, the growth of the pyramid is highly nonlinear. The reason comes from the fact that the behavior of each layer is extremely complicated (see [5, Section 5]). One particular layer can receive some amount of crystal in each nucleation step, then changes its shape in each propagation step. Of course the layers change not only in a nonlinear way but also in a nonhomogeneous way in each propagation step. Furthermore, the changes are not monotone (unlike the case $V = 1$). These affect the next nucleation step seriously as the receipt of crystals at each layer will change dramatically from time to time. More or less, this says that the problem has double nonlinear effects.

3. Lipschitz estimate

Let us first recall basic facts which are standard in the theory of viscosity solutions:

- For any given initial data u_0, which is bounded uniformly continuous, (C) has a viscosity solution $u \in C(\mathbb{R}^n \times [0,\infty))$ which is bounded in $\mathbb{R}^n \times [0,T]$ for each $T > 0$.
- The comparison principle for (C) holds.

See a monograph [3] for proofs for instance.

In this section, we establish a Lipschitz estimate with respect to x for solutions to (C), which is uniform on t.

Lemma 3.1. *For $u_0 \equiv 0$, let u be the corresponding solution to* (C). *There exists $R_0 > 0$ such that for each $T > 0$, we have*

$$u(x_T, s_T) = \max_{\mathbb{R}^n \times [0,T]} u \quad \textit{for some } (x_T, s_T) \in \overline{B}(0, R_0) \times [0,T].$$

Proof. If $f \equiv 0$, then $u \equiv 0$ and there is nothing to prove. We hence may assume that $f \not\equiv 0$. It is clear then that $u \geq 0$ and $u \not\equiv 0$.

Fix $T > 0$ and set $\sigma = \sup_{\mathbb{R}^n \times [0,T]} u > 0$. For $\varepsilon, \delta > 0$ sufficiently small, there exists $(x_{\varepsilon,\delta}, t_{\varepsilon,\delta}) \in \mathbb{R}^n \times (0,T]$ such that

$$u(x_{\varepsilon,\delta}, t_{\varepsilon,\delta}) = \max_{\mathbb{R}^n \times [0,T]} \left(u(x,t) - \varepsilon t - \delta(|x|^2+1)^{1/2}\right) > 0.$$

Set

$$F(p,X) := -\mathrm{tr}\left(\left(I_n - \frac{p \otimes p}{|p|^2}\right) X\right) - |p| \quad \text{for } (p,X) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n.$$

It is clear that F is degenerate elliptic and $F_*(0,0) = F^*(0,0) = 0$, where F_*, F^* denote half-relaxed limits (see [3] for definition).

By the definition of viscosity subsolution, we have

$$\varepsilon + F\left(\delta \frac{x_{\varepsilon,\delta}}{(|x_{\varepsilon,\delta}|^2+1)^{1/2}}, \delta \frac{(|x_{\varepsilon,\delta}|^2+1)I_n - x_{\varepsilon,\delta} \otimes x_{\varepsilon,\delta}}{(|x_{\varepsilon,\delta}|^2+1)^{3/2}}\right) \leq f(x_{\varepsilon,\delta}),$$

where I_n is the identity matrix of size n. Let $\delta \to 0$ first to deduce that $(x_{\varepsilon,\delta}, t_{\varepsilon,\delta}) \to (x_\varepsilon, t_\varepsilon)$ by passing to a subsequence if necessary and $x_\varepsilon \in \overline{B}(0, R_0)$ as $f = 0$ on $\mathbb{R}^n \setminus B(0, R_0)$. We then let $\varepsilon \to 0$ to get the desired result. $\square$

Lemma 3.2. *Let u be the solution to* (C) *with the initial data $u_0 \equiv 0$. Then*

$$\|u_t\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq M_f,$$

where $M_f = \max_{\mathbb{R}^n} f$.

Proof. It is clear that $\varphi(x,t) = M_f t$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ is a supersolution to (C) because of the fact that $F_*(0,0) = F^*(0,0) = 0$. We use the comparison principle to get

$$0 \leq u(x,t) \leq M_f t \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{3.1}$$

Thus, $\|u_t(\cdot,0)\|_{L^\infty(\mathbb{R}^n)} \leq M_f$.

For any given $s > 0$, both $(x,t) \mapsto u(x,t+s)$ and $(x,t) \mapsto u(x,t)$ are viscosity solutions to (C) with initial data $u(\cdot,s)$ and $u(\cdot,0)$, respectively. By the comparison principle and (3.1),

$$\|u(\cdot,t+s) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u(\cdot,s) - u(\cdot,0)\|_{L^\infty(\mathbb{R}^n)} \leq M_f s.$$

Divide both sides of the above by s and let $s \to 0$ to get the conclusion. ☐

Proposition 3.3. *Let u be the solution to* (C) *with given initial data $u_0 \equiv 0$. Then, there exists $C > 0$ such that*

$$\|Du\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq C.$$

Proof. For $\varepsilon \in (0,1)$, we consider the following approximated equation

$$\begin{cases} u_t^\varepsilon - \left(\operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{|Du^\varepsilon|^2+\varepsilon^2}}\right)+1\right)\sqrt{|Du^\varepsilon|^2+\varepsilon^2} - f = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{3.2}$$

This has a unique solution $u^\varepsilon \in C^2(\mathbb{R}^n \times [0,\infty))$. Setting $b^\varepsilon(p) := I_n - p \otimes p/(|p|^2+\varepsilon^2)$, we rewrite (3.2) as

$$u_t^\varepsilon - b_{ij}^\varepsilon(Du^\varepsilon)u_{x_ix_j}^\varepsilon - \sqrt{|Du^\varepsilon|^2+\varepsilon^2} - f = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{3.3}$$

Here we use Einstein's convention.

We use the Bernstein method to get the gradient bound for u^ε, hence u. Let $w^\varepsilon := |Du^\varepsilon|^2/2$. Differentiate the above equation with respect to x_k and multiply by $u_{x_k}^\varepsilon$ to yield

$$w_t^\varepsilon - b_{ij}^\varepsilon\left(w_{x_ix_j}^\varepsilon - u_{x_jx_k}^\varepsilon u_{x_ix_k}^\varepsilon\right) - Df\cdot Du^\varepsilon - u_{x_ix_j}^\varepsilon D_p b_{ij}^\varepsilon \cdot Dw^\varepsilon + \frac{Du^\varepsilon\cdot Dw^\varepsilon}{\sqrt{|Du^\varepsilon|^2+\varepsilon^2}} = 0.$$

Fix $T > 0$. Take $(x_0,t_0) \in \mathbb{R}^n \times (0,T]$ so that $w^\varepsilon(x_0,t_0) = \max_{\mathbb{R}^n\times[0,T]} w^\varepsilon$. At this point, we have

$$b_{ij}^\varepsilon u_{x_jx_k}^\varepsilon u_{x_ix_k}^\varepsilon - Df\cdot Du^\varepsilon \leq 0. \tag{3.4}$$

By using a modified Cauchy-Schwarz inequality

$$(\operatorname{tr} AB)^2 \leq \operatorname{tr}(ABB)\operatorname{tr} A \quad \text{for all } A, B \in \mathbb{S}^n,\ A \geq 0, \tag{3.5}$$

we obtain

$$Df\cdot Du^\varepsilon \geq \operatorname{tr}(b^\varepsilon(Du^\varepsilon)D^2u^\varepsilon D^2u^\varepsilon) \geq \frac{(\operatorname{tr}(b^\varepsilon(Du^\varepsilon)D^2u^\varepsilon))^2}{\operatorname{tr}(b^\varepsilon(Du^\varepsilon))} \geq \frac{(\operatorname{tr}(b^\varepsilon(Du^\varepsilon)D^2u^\varepsilon))^2}{n}. \tag{3.6}$$

By repeating the proof of Lemma 3.2, we have that $\|u_t^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq M_f + 1$ for all $\varepsilon \in (0,1)$. We use this and (3.3) to yield

$$\left(\operatorname{tr}\left(b^\varepsilon(Du^\varepsilon)D^2u^\varepsilon\right)\right)^2 = \left(u_t^\varepsilon - \sqrt{|Du^\varepsilon|^2+\varepsilon^2} - f\right)^2 \geq \frac{1}{2}|Du^\varepsilon|^2 - C, \tag{3.7}$$

where $C = (2M_f+1)^2$.

Combining (3.6) and (3.7) together, we obtain

$$\frac{1}{2}|Du^\varepsilon|^2 - C \leq nDf\cdot Du^\varepsilon \leq C|Du^\varepsilon|,$$

which implies that $\|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq C$ for some $C>0$ depending only on $\|f\|_{L^\infty}$, $\|Df\|_{L^\infty}$, and n. Let $\varepsilon \to 0$ to yield the desired result. □

Remark 1. It is worth emphasizing that the Lipschitz bound which is independent of t in Proposition 3.3 is essential to obtain the existence of the asymptotic speed. Indeed, the time local Lipschitz bound, which can be easily obtained by only using the comparison principle, is not enough.

4. Existence of Asymptotic Speed

Below is one of main results in [4].

Theorem 4.1. *Let u be the solution to* (C). *There exists $c \in [0, M_f]$ such that*

$$\lim_{t\to\infty}\frac{u(x,t)}{t} = c \qquad \textit{locally uniformly for } x \in \mathbb{R}^n. \tag{4.1}$$

Furthermore, c is independent of the choice of u_0.

Proof. Since the comparison principle holds, in order to prove (4.1), we can assume that $u_0 \equiv 0$. Recall that (3.1) gives us

$$0 = u_0(x) \leq u(x,t) \leq M_f t.$$

For $t \geq 0$, set $m(t) = \sup_{x\in\mathbb{R}^n} u(x,t)$. We now show that

$$m(t+s) \leq m(t) + m(s) \qquad \text{for all } s,t\geq 0. \tag{4.2}$$

Fix $s \geq 0$. We note that $(x,t) \mapsto v(x,t) = u(x,t+s) - m(s)$ and $(x,t)\mapsto u(x,t)$ are both solutions to (C), and

$$v(x,0) = u(x,s) - m(s) \leq 0 = u(x,0).$$

Thus, $v(x,t) \leq u(x,t)$ in light of the comparison principle. In particular, we get that (4.2) holds, which means that m is subadditive on $[0,\infty)$. By Fekete's lemma, there exists $c \in [0,\infty)$ such that

$$\lim_{t\to\infty}\frac{m(t)}{t} = c = \inf_{s>0}\frac{m(s)}{s}. \tag{4.3}$$

It is clear that $c \leq M$ because of (3.1). If $c=0$, then (4.1) holds immediately. We therefore only need to consider the case that $c>0$. Fix $\varepsilon>0$. There exists $T = T(\varepsilon) > 0$ such that

$$c \leq \frac{m(t)}{t} \leq c+\varepsilon \qquad \text{for all } t > T.$$

For $t > \max\{\frac{MT}{c}, M\}$, we use Lemma 3.1 to have that

$$ct \le \max_{\mathbb{R}^n\times[0,t]} u = u(x_t, s_t) \le M s_t \quad \text{for some } (x_t, s_t) \in \overline{B}(0, R_0) \times [0, t], \tag{4.4}$$

which implies that $s_t \ge \frac{ct}{M} \ge T$. Thus, we are able to improve (4.4) as

$$ct \le \max_{\mathbb{R}^n\times[0,t]} u = u(x_t, s_t) \le (c+\varepsilon)s_t, \tag{4.5}$$

which yields $s_t \ge \frac{c}{c+\varepsilon}t$. So for any $x \in B(0,R)$ for $R > 0$ given, we use Lemma 3.2 and Proposition 3.3 to estimate that

$$|u(x,t) - u(x_t, s_t)| \le C(|x - x_t| + |t - s_t|) \le C(R + R_0) + \frac{C\varepsilon t}{c+\varepsilon}.$$

Hence,

$$c - \frac{C\varepsilon}{c+\varepsilon} - \frac{C(R+R_0)}{t} \le \frac{u(x,t)}{t} \le c + \varepsilon.$$

The proof is complete. □

Remark 2. We note that the use of the Fekete lemma is quite natural in the literature once some subadditive quantities are identified. A related result in the periodic setting appeared in a lecture note by Barles [1, Theorem 10.2]. In general, the lack of periodicity prevents us from using the natural compactness property of $\mathbb{T}^n$. In a sense, (1.1) is a compactness assumption, which is a simple and effective replacement for the periodicity and also quite natural from viewpoint of physicists.

5. Qualitative properties on asymptotic speed

In this section, we investigate qualitative properties on asymptotic speed. Let u be the solution to (C) and we denote the asymptotic speed by c_f to emphasize the dependence on f.

5.1. **Radial symmetric case.** In this subsection, we assume a radially symmetric condition for f, i.e., $f(x) = \tilde{f}(|x|)$ for $x \in \mathbb{R}^n$, where $\tilde{f} : [0,\infty) \to \mathbb{R}$ is given. The following theorem gives a complete characterization of c_f.

Theorem 5.1. *Assume that $f(x) = \tilde{f}(|x|)$ for $x \in \mathbb{R}^n$, where $\tilde{f} \in C_c([0,\infty),[0,\infty)) \cap \mathrm{Lip}([0,\infty),[0,\infty))$. Let u be the solution to (C). Then,*

$$c_f = \max_{r\in[n-1,\infty)} \tilde{f}(r) = \max_{|x|\ge n-1} f(x). \tag{5.1}$$

In order to prove this theorem, we consider a radially symmetric solution $u(x,t) = \phi(|x|,t)$ with $u(x,0) = 0$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Then $\phi = \phi(r,t) : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfies the 1-dimensional Hamilton–Jacobi equation:

$$\begin{cases} \phi_t - \frac{n-1}{r}\phi_r - |\phi_r| = \tilde{f}(r) & \text{in } (0,\infty)\times(0,\infty), \\ \phi(\cdot,0) = 0 & \text{on } [0,\infty). \end{cases} \tag{5.2}$$

Lemma 5.2. *Set $u(x,t) := \tilde{\phi}(|x|,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, where $\tilde{\phi} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is the function defined by*

$$\tilde{\phi}(r,t) = \sup\left\{ \int_0^t \tilde{f}(\gamma(s))\,ds \;:\; \gamma([0,t]) \subset (0,\infty),\ \gamma(t) = r,\ \left|\gamma'(s) + \frac{n-1}{\gamma(s)}\right| \le 1 \ a.e. \right\}. \tag{5.3}$$

Then, u is the viscosity solution to (C).

Notice here that since we consider the viscosity solution (which may not be smooth at $x = 0$) of (C), we do not know the boundary condition of ϕ at $r = 0$ a priori. Therefore, it is not clear a priori that we have the representation for u in Lemma 5.3. We refer to [4] for a proof.

An important point in this representation is that we have the constraint

$$\left|\gamma'(s) + \frac{n-1}{\gamma(s)}\right| \le 1 \quad \text{for a.e. } s \in (0,t). \tag{5.4}$$

Therefore, for any $R \in (0, n-1)$, if γ is in the admissible class of (5.3) such that $\gamma(s) \in B(0,R)$, then

$$\gamma'(s) \le 1 - \frac{n-1}{\gamma(s)} \le 1 - \frac{n-1}{R} = -\frac{n-1-R}{R} =: -d < 0.$$

This says that the admissible trajectory cannot stay in $B(0,n-1)$ for a long time. From this observation, intuitively, we can see that the asymptotic speed is characterized by (5.1). We point out here that such a constraint which creates an interesting phenomenon from the view point of the asymptotic speed comes from the noncoercivity of Hamiltonian in (5.2).

We omit the detail of the proof of Theorem 5.1 and refer to [4].

Example 1. Set

$$f_r(x_1,x_2) := \max\{-(\sqrt{|x_1|^2 + |x_2|^2}) + r, 0\}$$

for $r \ge 0$. Set $c(r) := c_{f_r}$. Due to Theorem 5.1, we obtain

$$c(r) = \max\{0, r-1\} \quad \text{for all } r \ge 0.$$

5.2. **Non-radial symmetric case.** It non-radial symmetric settings, it seems to be much complicated and hard to obtain detailed qualitative properties of the asymptotic speed. We here give some of partial results.

By using the comparison principle and Theorem 5.1, we get several results for a general compact set E.

Proposition 5.3. *If $E \subset B(y, n-1)$ for some $y \in \mathbb{R}^n$, then*

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = 0 \quad \textit{uniformly for } x \in \mathbb{R}^n.$$

If $\overline{B}(y,n-1) \subset \operatorname{int} E$ for some $y \in \mathbb{R}^n$, then

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = c \quad \textit{locally uniformly for } x \in \mathbb{R}^n.$$

Proposition 5.4. *Assume there exist $s \in (0, M_f)$ and $R < n-1$ such that*

$$\{x \in \mathbb{R}^n : M_f - s \le f(x) \le M_f\} \subset B(0, R).$$

Let u be the solution to (C). *Then $c_f \le M - s$.*

Proposition 5.5. *Assume that there exists $R \ge n-1$ such that*

$$B(0, R) \subset \{x \in \mathbb{R}^n : f(x) = M_f\}$$

Let u be the solution to (C). Then $c_f = M_f$.

See [5, 4] for proofs of Propositions 5.3–5.5.

Example 2. Let $R_0 \in (1/2, 1)$. Set

$$f_a(x) := \max\{-|(x_1, x_2) - (a, 0)| + R_0, -|(x_1, x_2) - (-a, 0)| + R_0, 0\}$$

for $a \ge 0$. Set $c(a) := c_{f_a}$. By Proposition 5.3, we have

$$c(a) = 0 \quad \text{if } 0 < a < 1 - R_0, \quad c(a) = 0 \quad \text{if } a > R_0.$$

We can easily prove that $a \mapsto c(a)$ is continuous by using the stability of viscosity solutions, and know that $c(a) > 0$ for all $a \in (1 - R_0, 1)$ due to [5, Theorem 5.6]. However, we still do not precisely know the shape of $c(a)$. For instance, it is not clear yet which a gives the maximum of $c(a)$, which seems to be important from view point of the theory of crystal growth.

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References

1. G. Barles, An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications. Hamilton-Jacobi equations: approximations, numerical analysis and applications, 49–109, Lecture Notes in Math., 2074, Fond. CIME/CIME Found. Subser., Springer, Heidelberg, 2013.
2. G. Barles, P. E. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal. 4 (1991), no. 3, 271–283.
3. Y. Giga, *Surface Evolution Equations. A Level Set Approach*, Monographs in Mathematics, 99. Birkhäuser, Basel-Boston-Berlin (2006), xii+264pp.
4. Y. Giga, H. Mitake, T. Ohtsuka, H. V. Tran, *Existence of asymptotic speed of solutions to birth and spread type nonlinear partial differential equations*, in preparation.
5. Y. Giga, H. Mitake, H. V. Tran, *On asymptotic speed of solutions to level-set mean curvature flow equations with driving and source terms*, SIAM J. Appl. Math. 48 (2016), no. 5, 3515–3546.
6. M. Ohara and R. C. Reid, Modeling crystal growth rates from solution. *Prentice-Hall, Englewood Cliffs, N. J.* (1973).

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